Section 3.3 (cont'd)

recall: any basis of $\mathbb{R}^n$ consist of n vectors

ex. std basis of $\mathbb{R}^n$ is $\vec{e}_1, \dots, \vec{e}_n$

How to tell if n vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$ form a basis?

know $\vec{v}_1, \dots, \vec{v}_n$ form a basis of $\mathbb{R}^n$ iff every $\vec{b} \in \mathbb{R}^n$ can be written as LC of $\vec{v}_1, \dots, \vec{v}_n$

$$\vec{b} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \underbrace{\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{b}}_{\text{has a unique sol'n iff } A \text{ is invertible}}$$

ex. For which values of k do $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$ form a basis of $\mathbb{R}^3$?

write as a matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1-k}{2} \\ 0 & 0 & k^2-1 \end{bmatrix}$$

will become I_3 as long as $k^2 - 1 \neq 0$

$k \neq \pm 1$

conclusion: the 3 vectors in $\mathbb{R}^3$ form a basis iff k is neither -1 nor 1

§3.4 Coordinates

previously, used Cartesian coordinates on both
- xy-plane $\mathbb{R}^2$
- xyz-space $\mathbb{R}^3$

ex. given $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$ AND

plane $V := \text{span}(\vec{v}_1, \vec{v}_2)$ in $\mathbb{R}^3$

Q: is $\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ on V?

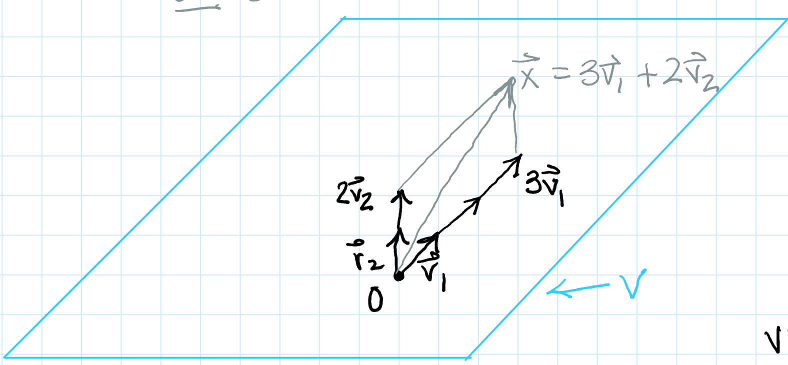
technique: determine if $\exists\, c_1, c_2$ s.t. $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$
using augmented matrix:

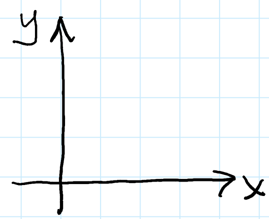
$$M = \left[\begin{array}{cc:c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right] \xrightarrow{\text{rref}} \text{rref}(M) = \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}$$

consistent

$\therefore\ \vec{x} = 3\vec{v}_1 + 2\vec{v}_2 \Rightarrow \vec{x}$ is on V

can be written as LC



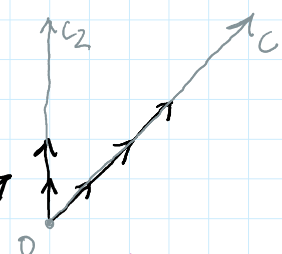


visualize this as a coordinate grid

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

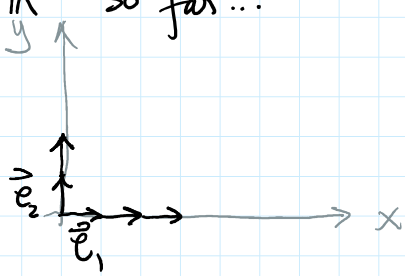
"address" on $c_1 - c_2$ coordinate system

coordinate vector

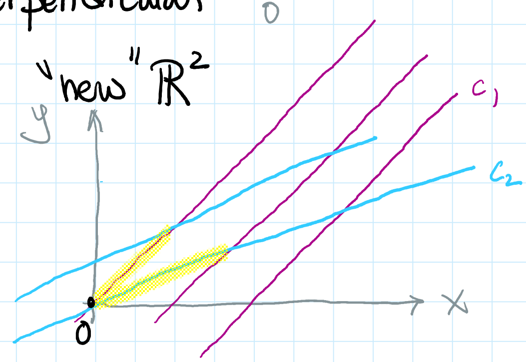


note: irrelevant that axes are not perpendicular

$\mathbb{R}^2$ so far...



"new" $\mathbb{R}^2$



notation for coordinate vector:
use "old English" font: $\mathfrak{B}$

let $\mathfrak{B}$ = basis of $\vec{v}_1, \vec{v}_2$ of V

then coordinate vector of $\vec{x}$ WRT to $\mathfrak{B}$ is denoted by $[\vec{x}]$

let $\mathcal{B}$ = basis of v_1, $\vec{v}_2$ of V

then coordinate vector of $\vec{x}$ WRT to $\mathcal{B}$ is denoted by $[\vec{x}]_{\mathcal{B}}$

from previous example, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{matrix} c_1 \\ c_2 \end{matrix}$

Generalize coordinates in $\mathbb{R}^n$

given basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)$ of subspace V of $\mathbb{R}^n$

then any vector $\vec{x} \in V$ can be uniquely written as:

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_m \vec{v}_m$$

where scalars $c_1, c_2, \dots, c_m$ are $\mathcal{B}$-coordinates of $\vec{x}$

and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$ is $\mathcal{B}$-coordinate vector of $\vec{x}$

likewise, can say that $\vec{x} = S[\vec{x}]_{\mathcal{B}}$ where $S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$

S is an $n \times m$ matrix

using previous examples: $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

verify this holds in $\vec{x} = S[\vec{x}]_{\mathcal{B}}$

$$\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\stackrel{?}{=} \begin{bmatrix} 1(3) + 1(2) \\ 1(3) + 2(2) \\ 1(3) + 3(2) \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \checkmark$$

thm: if $\mathcal{B}$ is a basis of subspace V of $\mathbb{R}^n$ then;

- $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$
- $[k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}}$

for $\vec{x}, \vec{y} \in V$ and all scalars k

ex. given basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ in $\mathbb{R}^2$ where $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

a. given $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ find $[\vec{x}]_{\mathfrak{B}}$

use $\vec{x} = S[\vec{x}]_{\mathfrak{B}}$

$\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} [\vec{x}]_{\mathfrak{B}}$ ← solve for $[\vec{x}]_{\mathfrak{B}}$

$S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ then $S^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad$ det(S)

$= \frac{1}{3(3)+1(1)}$

$S^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

$[\vec{x}]_{\mathfrak{B}} = S^{-1}[\vec{x}]$

$= \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+1 \\ -1+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{matrix} c_1 \\ c_2 \end{matrix}$ ← $[\vec{x}]_{\mathfrak{B}}$

b. given $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ find $\vec{y}$

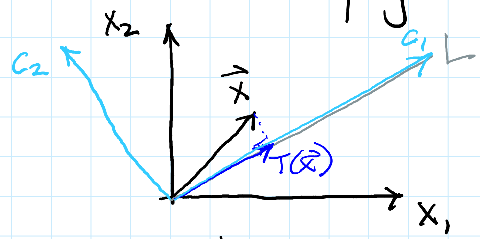
use $\vec{y} = S[\vec{y}]_{\mathfrak{B}}$

$\vec{y} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \vec{y}$

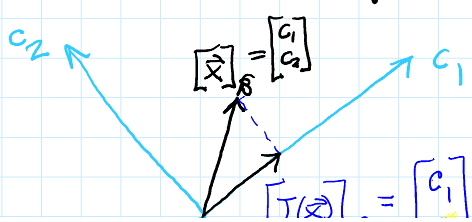
apply concept of coordinates to linear transformations :

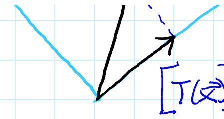
let line L in $\mathbb{R}^2$ be spanned by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

let $T : \mathbb{R}^2 \to \mathbb{R}^2$ projects any $\vec{x}$ orthogonally onto L



introduce a coordinate system s.t. L is an axis : c_1-axis
$L_\perp$ is c_2-axis





$[T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

then T is represented by $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ since $\begin{bmatrix} c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

↳ transforms $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ to $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$

apply this transformation, T, to previous example $\mathcal{B} = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$

find relationship between $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and A s.t. $T(\vec{x}) = A\vec{x}$

$\vec{x} = S[\vec{x}]_{\mathcal{B}}$ where $S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

$\vec{x} \xrightarrow{A} T(\vec{x})$ $\quad [\vec{x}]_{\mathcal{B}} \xrightarrow{B} [T(\vec{x})]_{\mathcal{B}}$ $\quad$ put these together:

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] & T(\vec{x}) \\
{[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S"']
\end{tikzcd}$$

i.e., $T(\vec{x}) = \underbrace{A\, S[\vec{x}]_{\mathcal{B}}}_{\text{apply } 1^{ST}}$

i.e. 1^{st} apply S to $[\vec{x}]_{\mathcal{B}}$ then apply A
the result will be $T(\vec{x})$

also, $T(\vec{x}) = S\, B[\vec{x}]_{\mathcal{B}}$

$\therefore\ A\, S[\vec{x}]_{\mathcal{B}} = S\, B[\vec{x}]_{\mathcal{B}}$ for $\forall \vec{x} \in \mathbb{R}^2$

OR

$AS = SB$ $\quad$ simplified version

↓

$S^{-1}AS = B$ $\quad$ solve for B $\quad$ $S^{-1}AS = \underbrace{S^{-1}S}_{I_n}B$

also $A = SBS^{-1}$ $\quad$ solve for A $\quad$ $A\underbrace{SS^{-1}}_{I_n} = SBS^{-1}$

↓ $\quad \begin{bmatrix} 3 & -1 \\ \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \end{bmatrix}\frac{1}{\ } \begin{bmatrix} 3 & 1 \\ \end{bmatrix}$

solve for A $\quad A\underbrace{SS^{-1}}_{I_n} = SDS^{-1}$

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{10}\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} .9 & .3 \\ .3 & .1 \end{bmatrix} = A$$

Similar Matrices

defn: given $n \times n$ matrices, A, B

A is similar to B if $\exists$ invertible matrix S s.t. $AS = SB$

or

$B = S^{-1}AS$

ex is $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ similar to $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$?

want $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ s.t. $AS = SB$ holds

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x+2z & y+2t \\ 4x+3z & 4y+3t \end{bmatrix} = \begin{bmatrix} 5x & -y \\ 5z & -t \end{bmatrix}$$

solve ex. $x+2z = 5x$

then $2x = z$

$t = -y$

$\Rightarrow S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$ $\qquad \det(S) = -xy - 2xy$

$= -3xy$

recall that S is invertible if $\det(S) \neq 0$ $\qquad x \neq 0,\ y \neq 0$

if choose $x = y = 1$, then $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

$\therefore$ A is similar to B